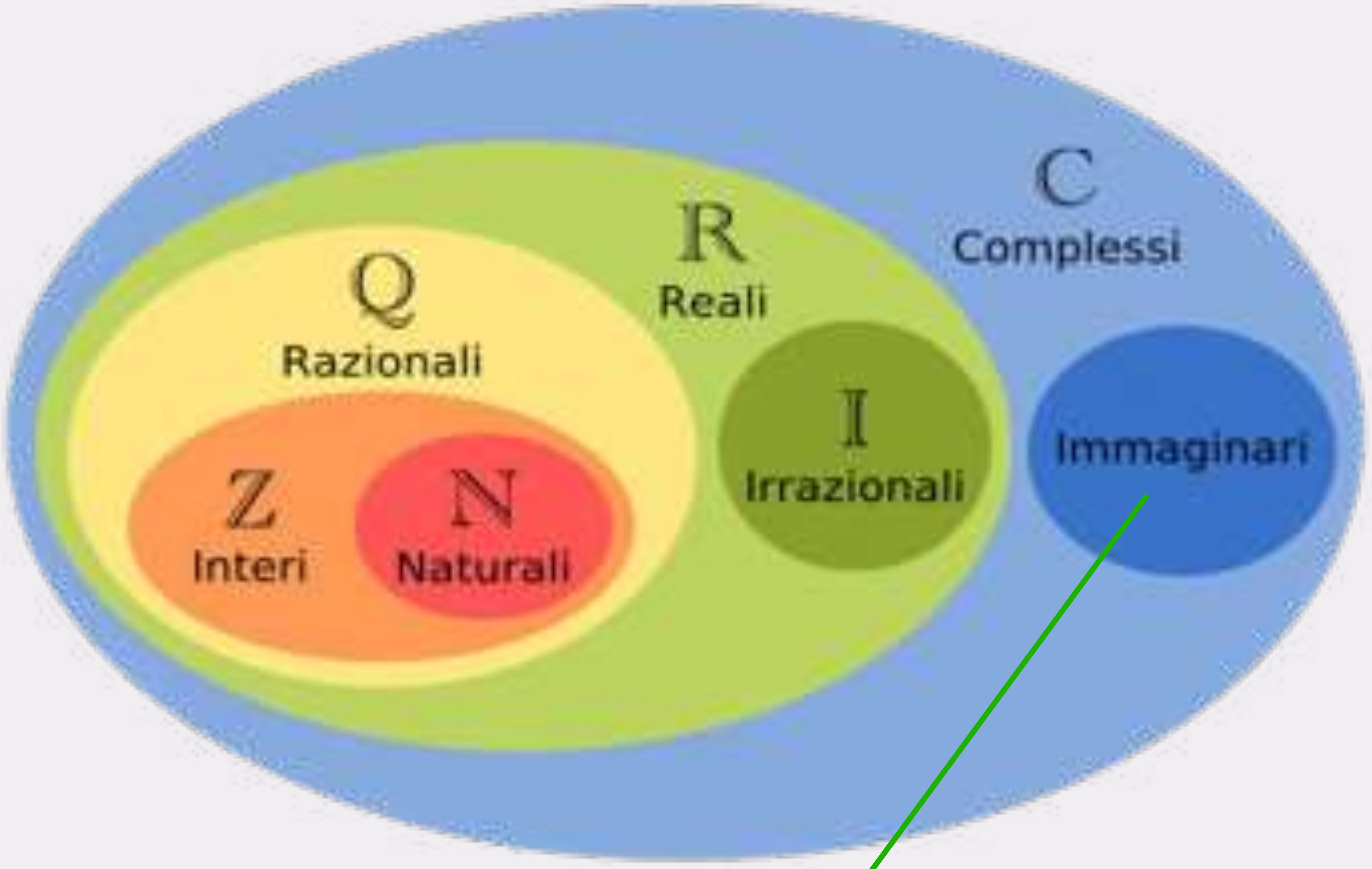
# i NUMERI COMPLESSI

# La **necessità** dei **numeri immaginari**




$2i \qquad -\sqrt{5}i \qquad \frac{3}{4}i$

$$x+5=0 \longrightarrow x=-5$$

$$2x-3=0 \longrightarrow x=\frac{3}{2}$$

$$x^2-2=0 \longrightarrow x=\pm\sqrt{2}$$

$$x^2+1=0 \longrightarrow x^2=-1$$

$$(\Delta<0)$$

$$\downarrow$$

$$x=\pm\sqrt{-1}$$

$$\downarrow$$

$$x=\pm i$$

$i$ = unità immaginaria

$$i^2=-1$$

**ESEMPI**

1. $x^2 + 9 = 0$

$x^2 = -9$

$x = \pm\sqrt{-9} = \pm 3i$ → Numeri immaginari

2. $x^2 - 2x + 2 = 0$

$\Delta = b^2 - 2ac = (-2)^2 - 4(1)(2) = -4$

$\Delta < 0$ : NO soluzioni reali

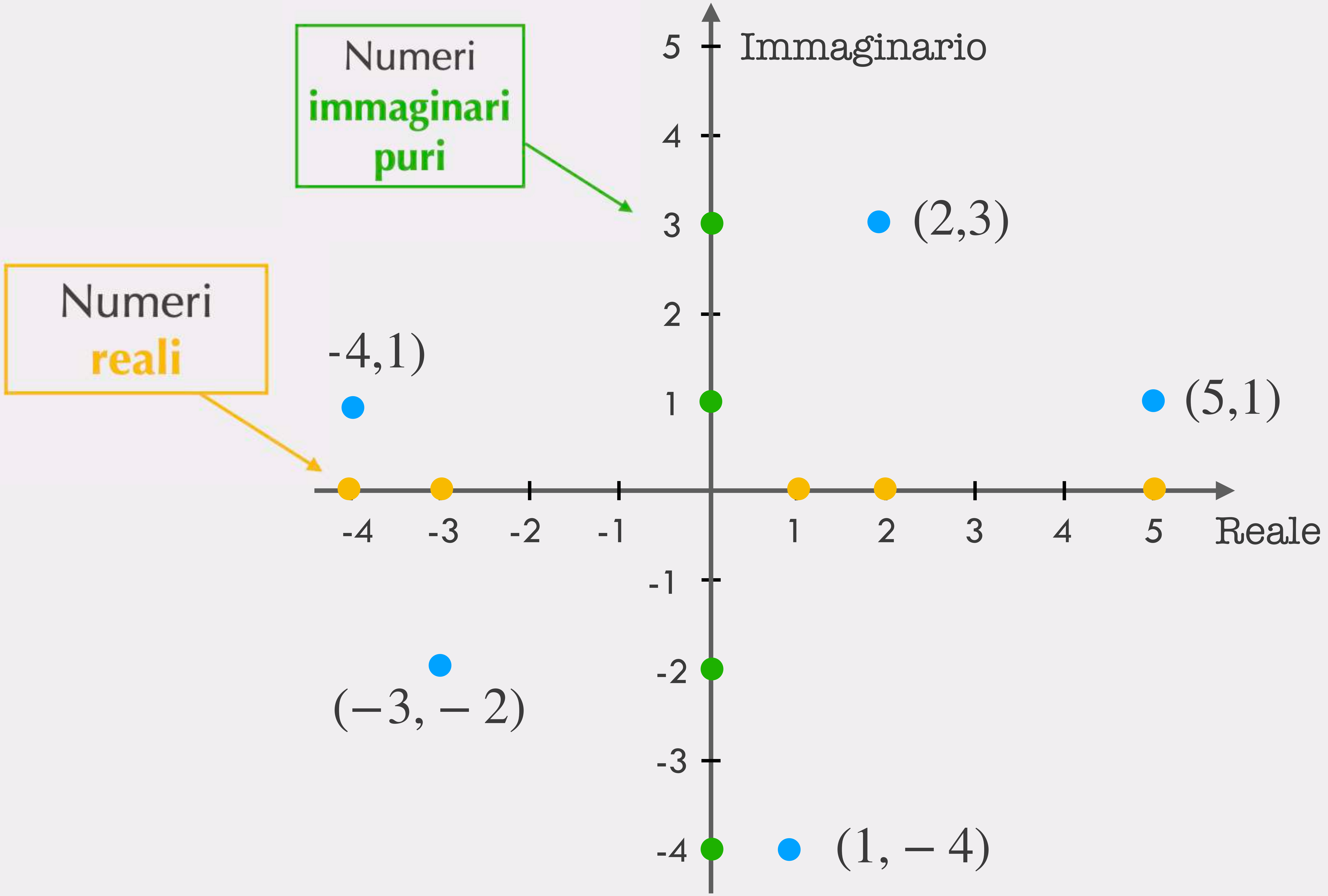
$$x_{1/2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$x_1 = 1 + i$

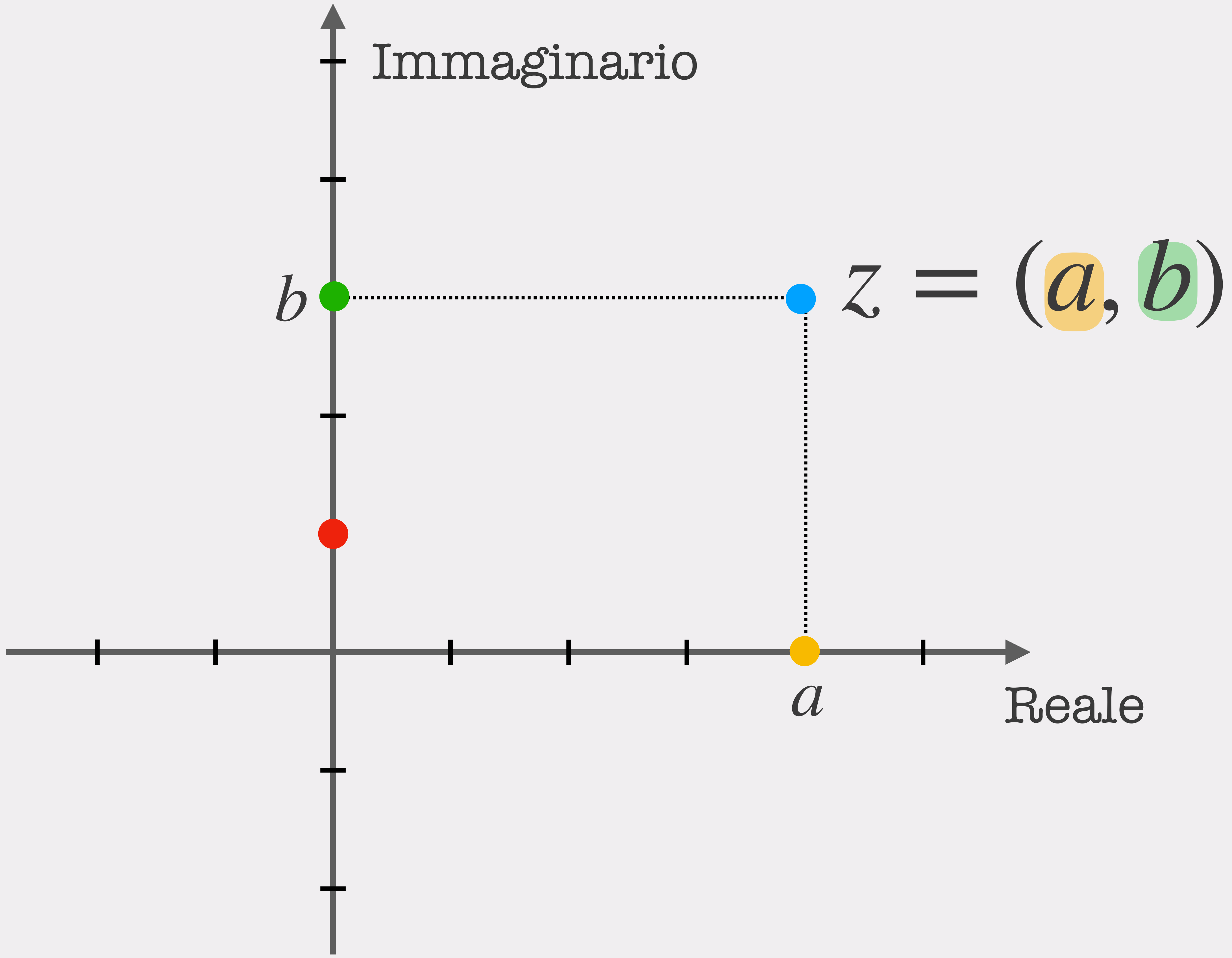
$x_2 = 1 - i$

→ Numeri complessi

# Numero complesso: $z$



# Coordinate **cartesiane**

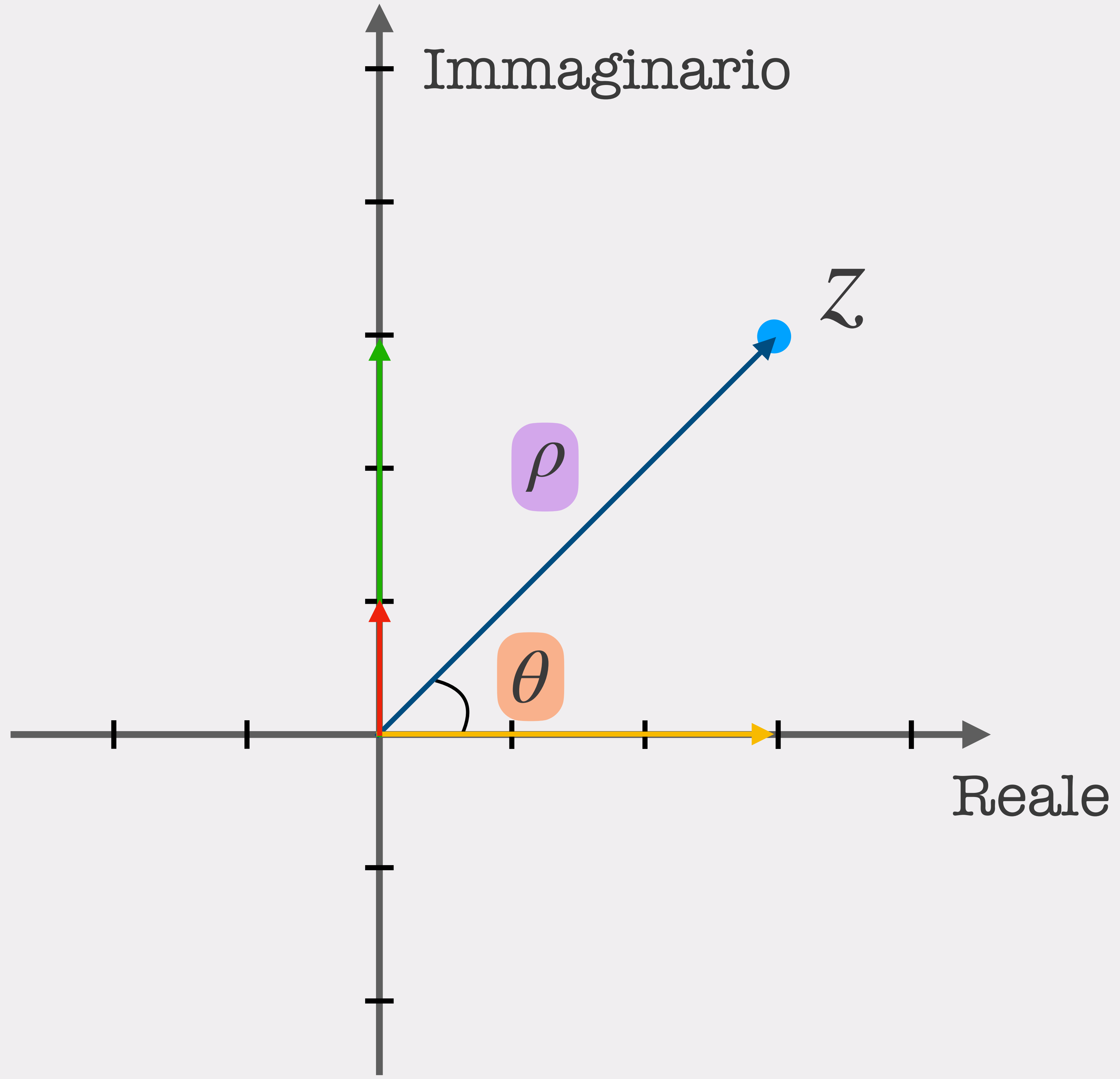



**Numero complesso:**
$z = (a, b)$

$a = \mathbf{Re}(z)$
$b = \mathbf{Im}(z)$

- $(a,0) = a \longrightarrow$ numero reale
- $(0,b) = bi \longrightarrow$ numero immaginario $\longrightarrow i = (0,1)$

# Coordinate **polari**




**Numero complesso:** $z$

$\rho$ e $\theta$ → coordinate polari del punto

$\rho$ e $\theta$ → $\begin{cases} \rho \geq 0 \\ 0 \leq \theta < 2\pi \end{cases}$

**RADIANTI**

- $\rho \ , \theta = 0 \ ; \ \pi$ ⟶ numero reale
- $\rho \ , \theta = \dfrac{\pi}{2} \ ; \ \dfrac{3}{2}\pi$ ⟶ numero immaginario
- $i \ \rightarrow \ \rho = 1 \ , \theta = \dfrac{\pi}{2}$

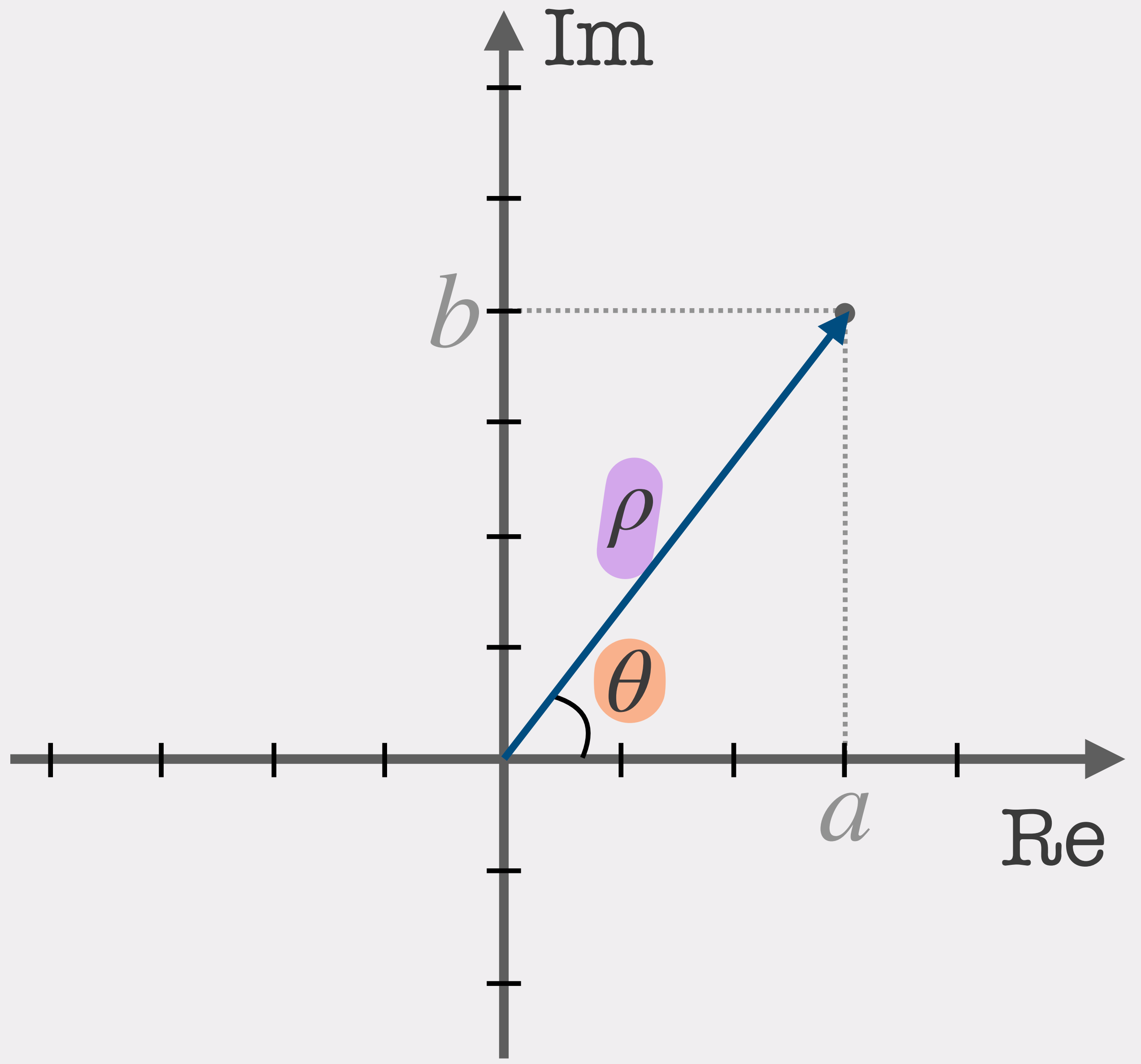
$\rho = |z|$ ⟶ modulo

$\theta = arg(z)$ ⟶ argomento

## Dalle **coordinate polari** alle **coordinate cartesiane**

$\rho, \theta \longrightarrow a, b$

- $a = \rho \cos(\theta)$
- $b = \rho \sin(\theta)$




## Dalle **coordinate cartesiane** alle **coordinate polari**

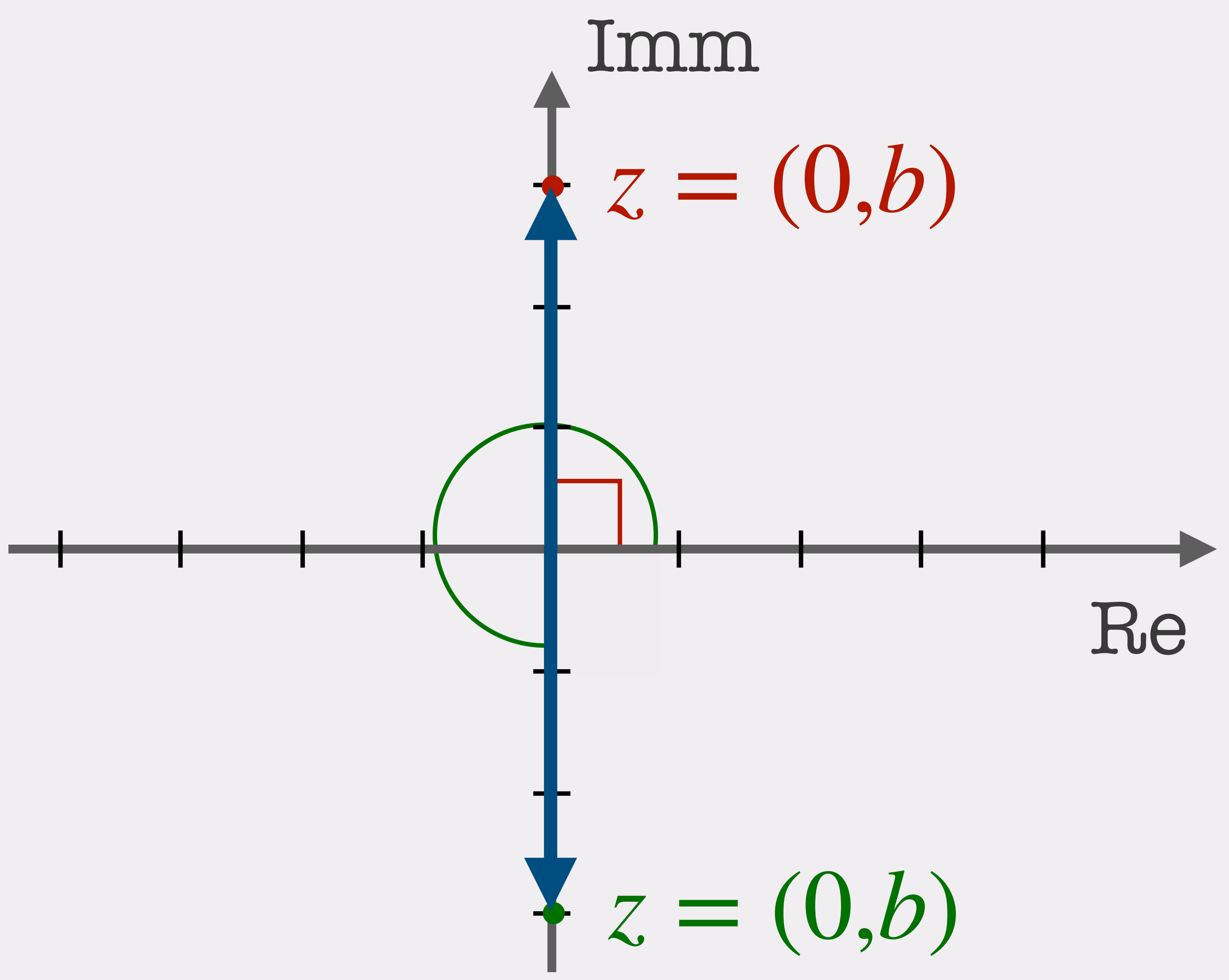
$a, b \longrightarrow \rho, \theta$

- $\rho = \sqrt{a^2 + b^2} \longrightarrow |z| = \sqrt{[\mathbf{Re}(z)]^2 + [\mathbf{Im}(z)]^2}$
- $\theta \longrightarrow$ $a = \rho\cos(\theta)$, $b = \rho\sin(\theta)$ $\longrightarrow$ $\rho = \dfrac{a}{\cos(\theta)}$, $\rho = \dfrac{b}{\sin(\theta)}$ $\longrightarrow \dfrac{a}{\cos(\theta)} = \dfrac{b}{\sin(\theta)}$

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{b}{a} \longrightarrow \tan\theta = \frac{b}{a}$$

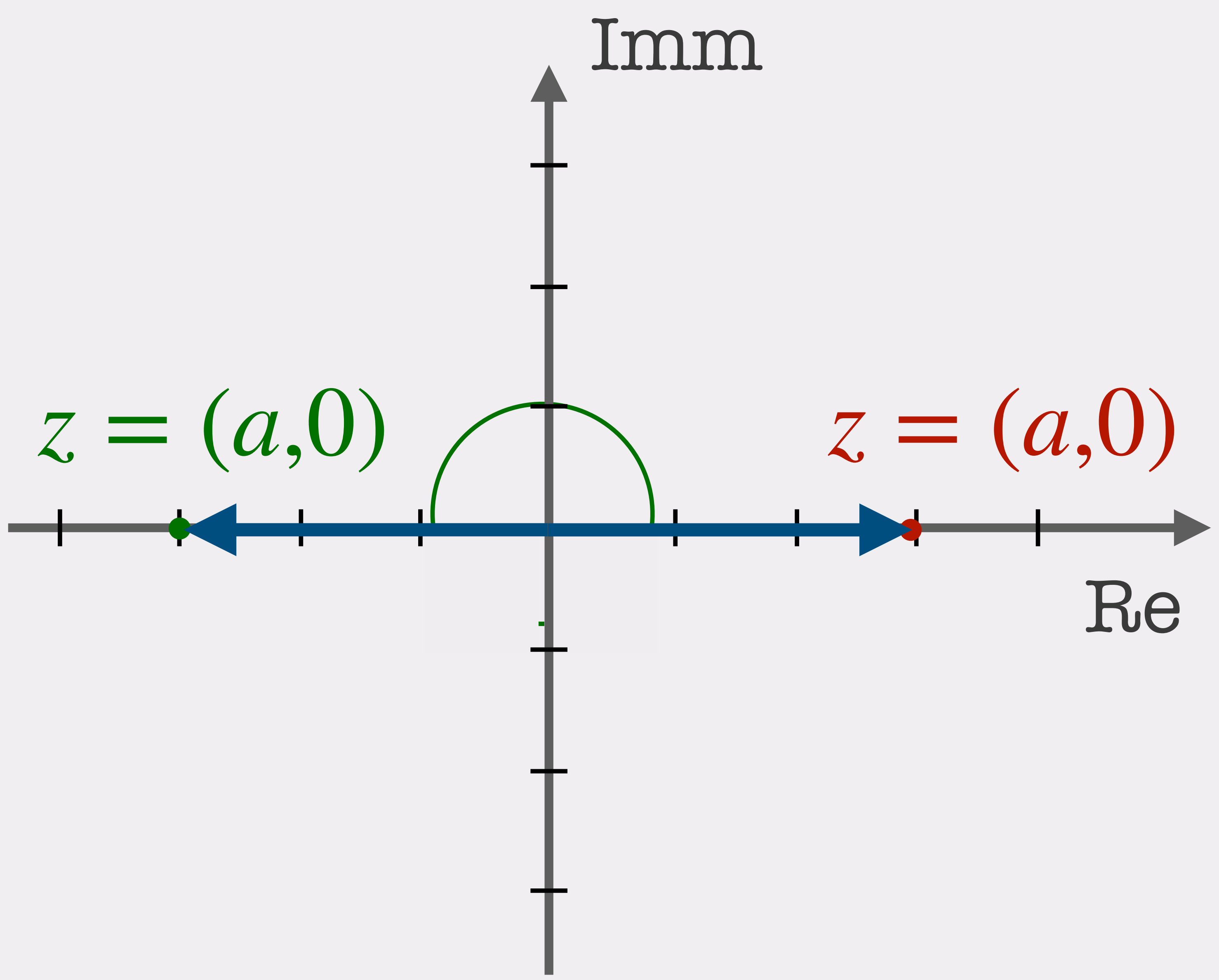
$a = 0$

| | |
|---|---|
| $b > 0 \longrightarrow$ | $\theta = \dfrac{\pi}{2}$ |
| $b < 0 \longrightarrow$ | $\theta = \dfrac{3}{2}\pi$ |
| $b = 0 \longrightarrow$ | $\rho = 0$<br>$\theta$ non è definito |



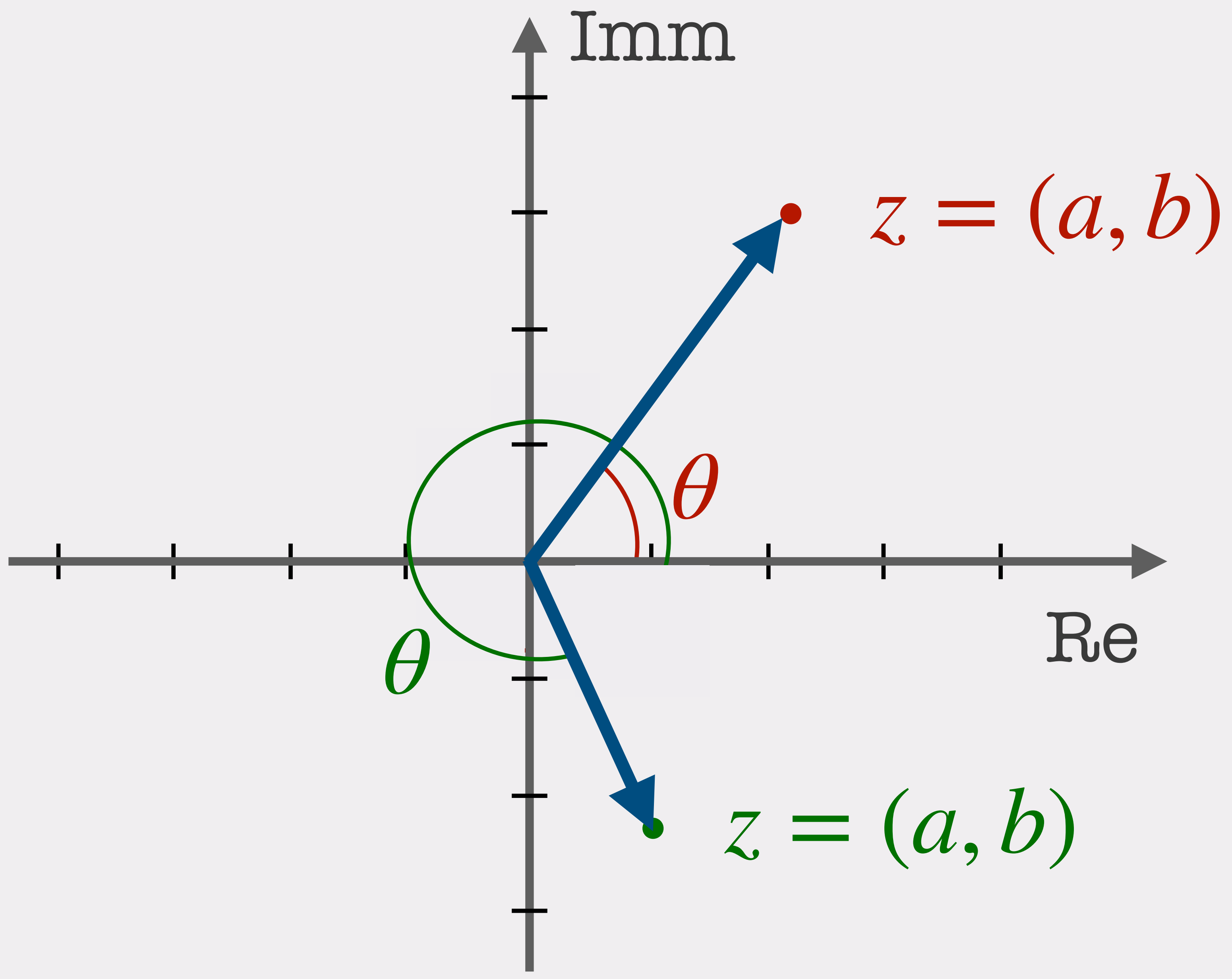

$b = 0$

| | |
|---|---|
| $a > 0 \longrightarrow$ | $\theta = 0$ |
| $a < 0 \longrightarrow$ | $\theta = \pi$ |



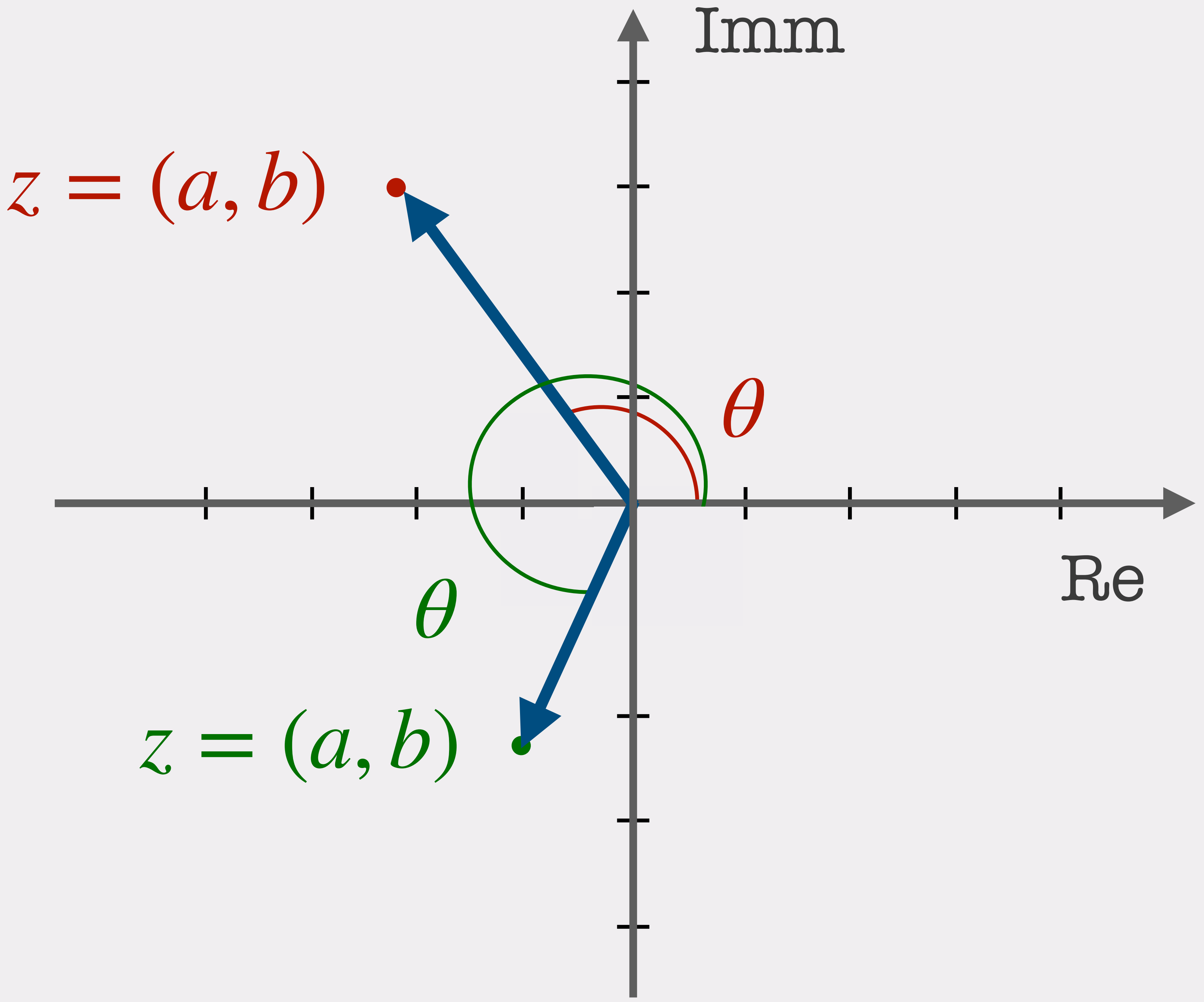
$a > 0$

| | |
|---|---|
| $b > 0 \longrightarrow \theta = \arctan\left(\frac{b}{a}\right)$ | |
| $b < 0 \longrightarrow \theta = \arctan\left(\frac{b}{a}\right) + 2\pi$ | |




$a < 0$

$b$ qualsiasi $\longrightarrow \theta = \arctan\left(\frac{b}{a}\right) + \pi$



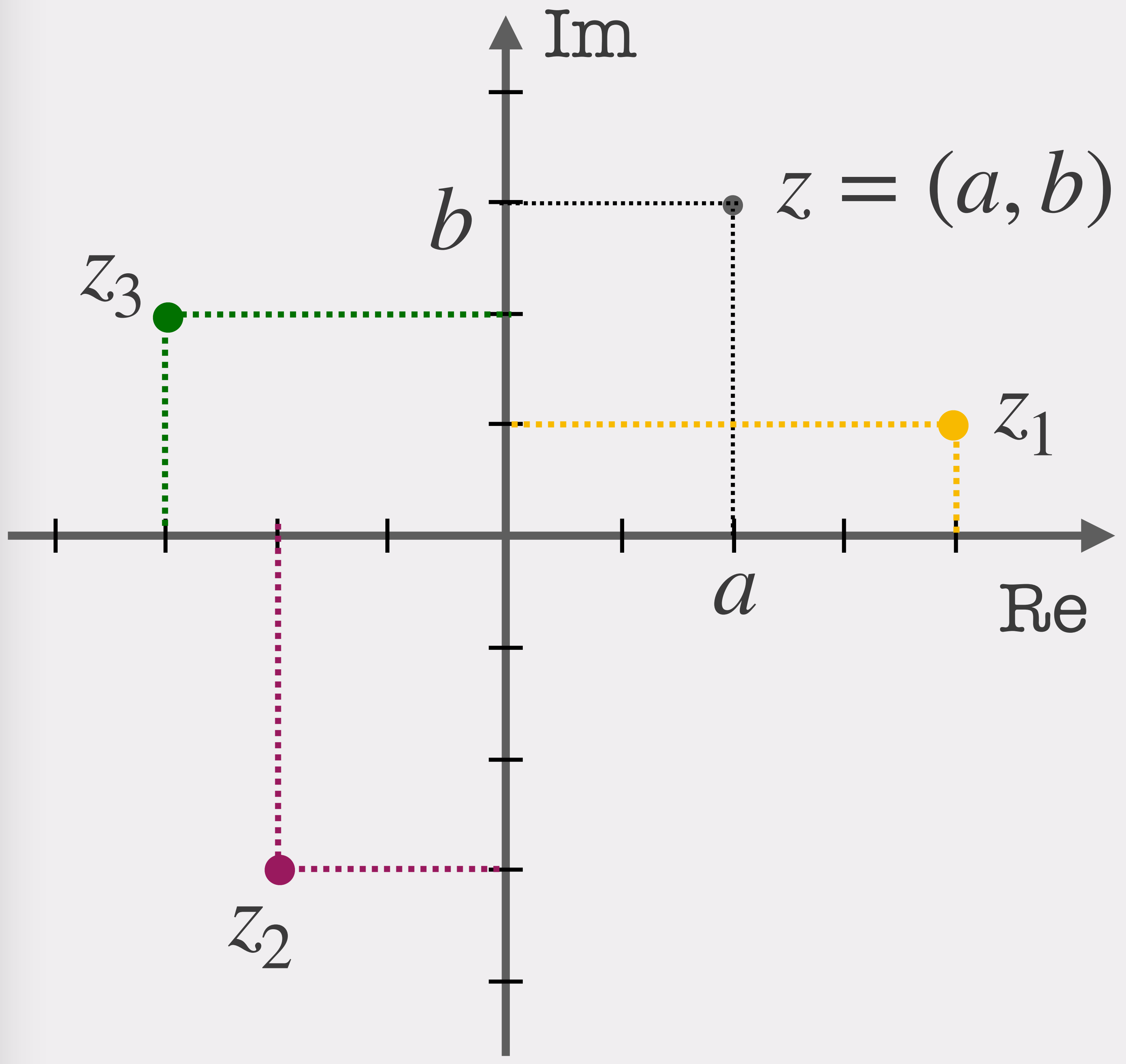
Esistono **3 forme** per esprimere un numero complesso

Coordinate **cartesiane** → 1 Forma **algebrica** (o forma **cartesiana**)

Coordinate **polari** → 2 Forma **trigonometrica** (o forma **polare**)

→ 3 Forma **esponenziale**

# Forma **algebrica** (o forma **cartesiana**)




$a$ e $b$ ⟶ coordinate cartesiane del punto

$$z = a + i \cdot b$$

$$a = \mathbf{Re}(z)$$
$$b = \mathbf{Im}(z)$$

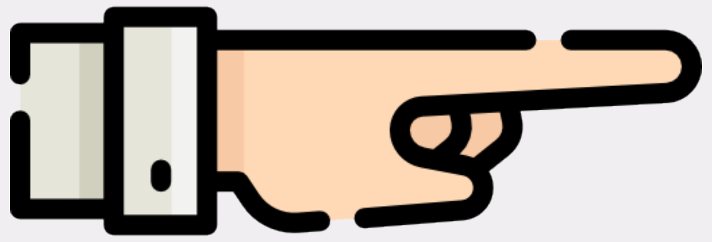
- $z_1 = 4 + i$
- $z_2 = -2 - 3i$
- $z_3 = -3 + 2i$

Calcolare il **MODULO**

$$|z| = \sqrt{[\mathbf{Re}(z)]^2 + [\mathbf{Im}(z)]^2}$$

1. $z_1 = 6 + 2i \longrightarrow |z_1| = \sqrt{a^2 + b^2} = \sqrt{(6)^2 + (2)^2} = \sqrt{36 + 4} = 2\sqrt{10}$

2. $z_2 = -3 + i \longrightarrow |z_2| = \sqrt{a^2 + b^2} = \sqrt{(-3)^2 + (1)^2} = \sqrt{9 + 1} = \sqrt{10}$

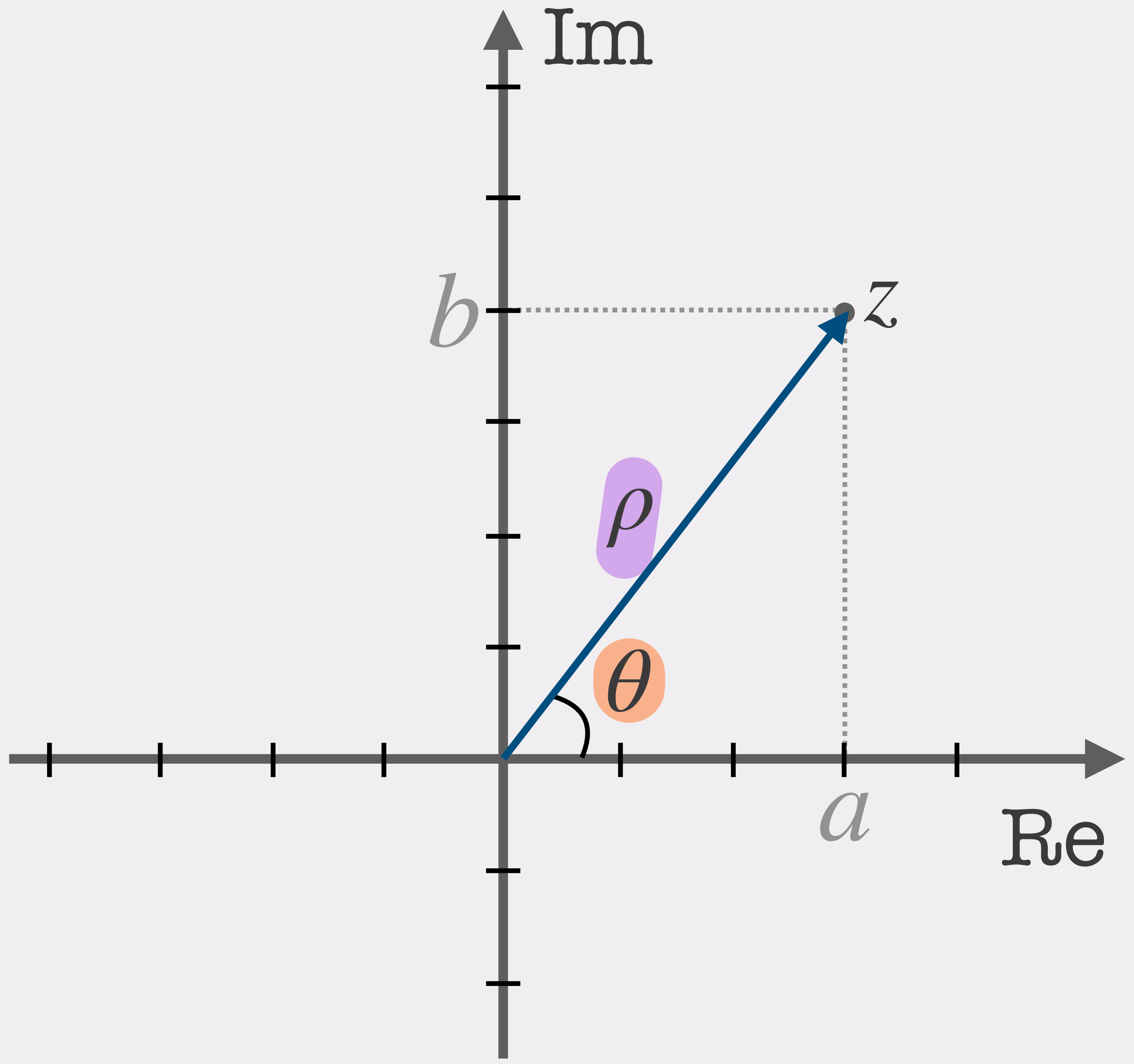


PROVA TU!

3. $z_3 = 3 - i$

4. $z_4 = -2 + 2i$

# Forma **trigonometrica** (o forma **polare**)




$\rho$ e $\theta$ → coordinate polari del punto

$\rho$ e $\theta$ → $\begin{cases} \rho \geq 0 \\ 0 \leq \theta < 2\pi \end{cases}$

$\rho = |z|$ → modulo

$\theta = arg(z)$ → argomento

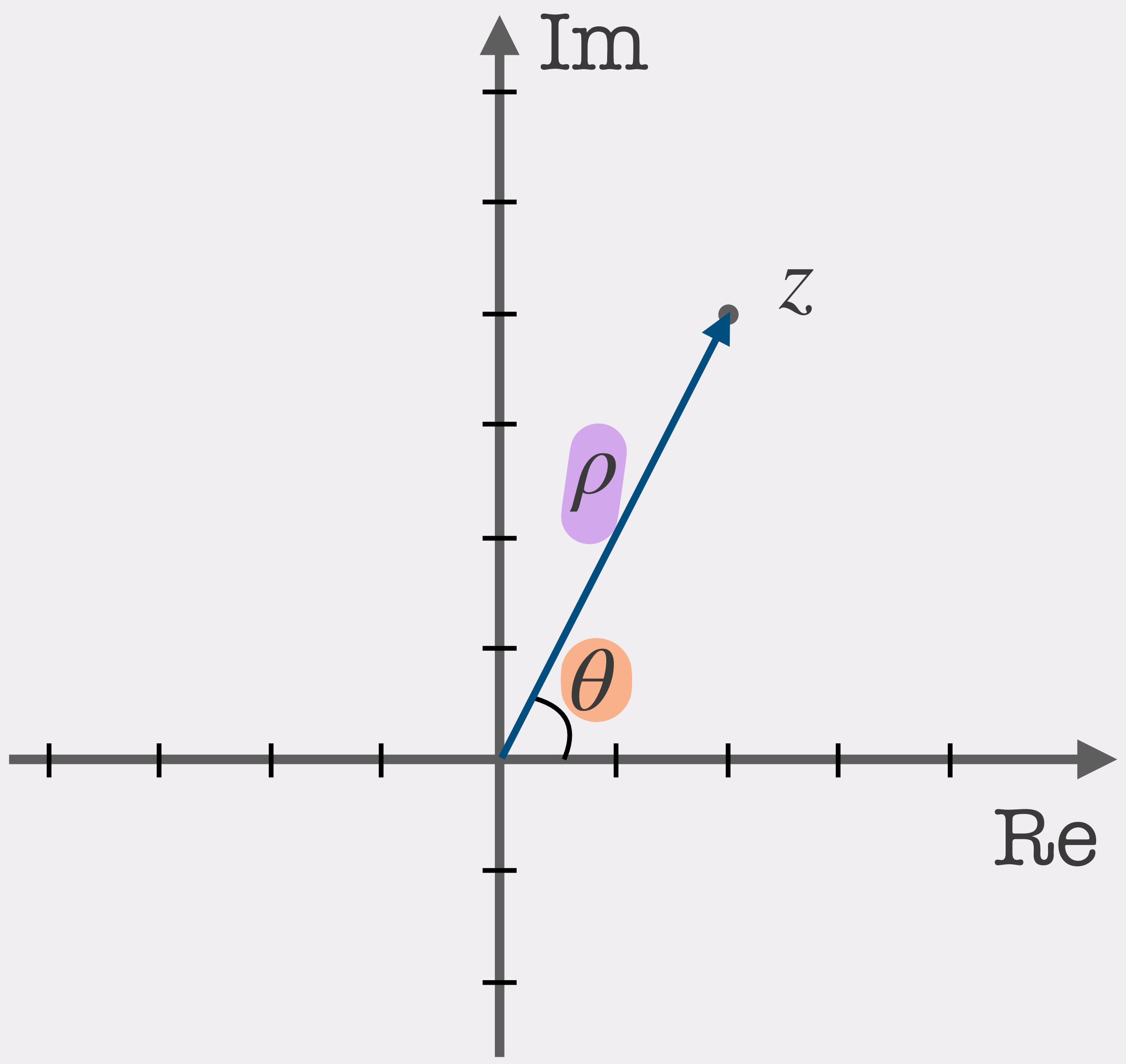
$$z = \rho\left[\cos(\theta) + i\sin(\theta)\right]$$

→ $z = a + ib$, con $a \to \rho\cos(\theta)$ e $b \to \rho\sin(\theta)$ ✓

$$z_1 = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

$$z_2 = \sqrt{3}\left[\cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right)\right]$$

# Forma **esponenziale**




$\rho$ e $\theta$ → coordinate polari del punto

$\rho$ e $\theta$ → $\begin{cases} \rho \geq 0 \\ 0 \leq \theta < 2\pi \end{cases}$

$\rho = |z|$ → modulo

$\theta = arg(z)$ → argomento

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

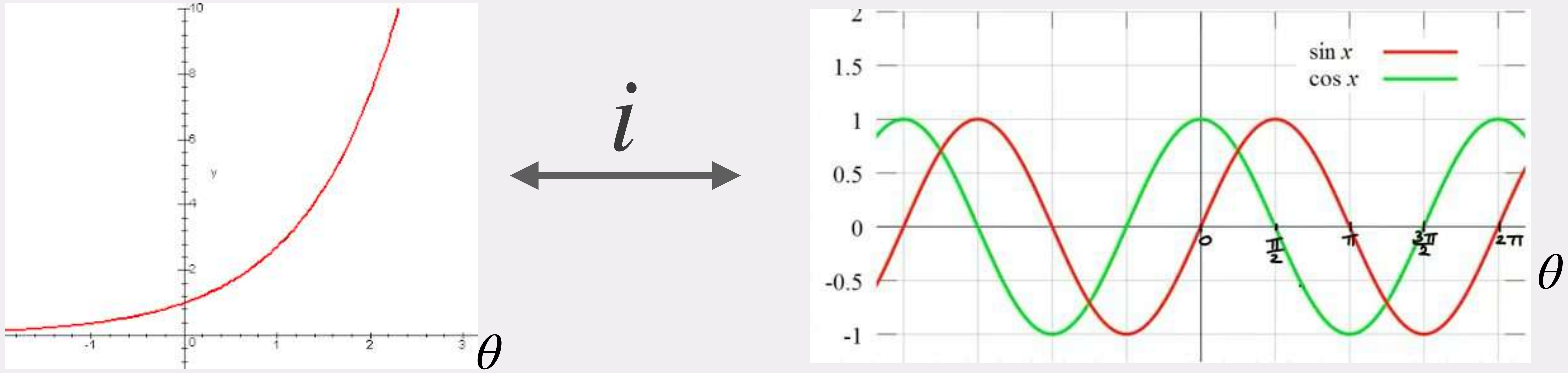
**FORMULA DI EULERO**

$$z = \rho\left[\cos(\theta) + i\sin(\theta)\right] \longrightarrow z = \rho e^{i\theta}$$

$$z_1 = \frac{1}{4}e^{i\pi}$$

$$z_2 = 3e^{i\frac{5}{6}\pi}$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$




$$\left(e^{i\theta}\right)^2 = e^{i2\theta} = \cos(2\theta) + i\sin(2\theta)$$

$$\downarrow$$

$$\left[\cos(\theta) + i\sin(\theta)\right]^2 = \cos(\theta)^2 + i^2\sin(\theta)^2 + 2i\cos(\theta)\sin(\theta)$$

$$= \cos(\theta)^2 - \sin(\theta)^2 + 2i\cos(\theta)\sin(\theta)$$

$$\cos(\theta)^2 - \sin(\theta)^2 \rightarrow \cos(2\theta) \qquad \cos(\theta)\sin(\theta) \rightarrow \sin(2\theta)$$

**SE:** $\theta = \pi \longrightarrow e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 \longrightarrow$

**IDENTITÀ DI EULERO**

$$e^{i\pi} + 1 = 0$$

# Dalla **forma trigonometrica** alla **forma esponenziale** e **viceversa**

$\rho, \theta$ $\rho, \theta$

$$z = \rho\left[\cos(\theta) + i\sin(\theta)\right]$$

$$z = \rho e^{i\theta}$$

1. $$z_1 = \frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] \longrightarrow z_1 = \frac{1}{2}\, e^{i\frac{\pi}{2}}$$

2. $$z_2 = \sqrt{2}\, e^{i\frac{\pi}{3}} \longrightarrow z_2 = \sqrt{2}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

PROVA TU!

3. $$z_3 = \frac{1}{4}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

4. $$z_4 = \sqrt{5}\, e^{i\pi}$$

# Dalle **forme trigonometrica** ed **esponenziale** alla **forma algebrica**

$\rho, \theta$ $\quad\quad\quad\quad$ $a, b$

1. $$z_1 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

- $a = \rho\cos(\theta) \longrightarrow a = 2\cos\left(\frac{\pi}{4}\right) = 2\frac{\sqrt{2}}{2} = \sqrt{2}$
- $b = \rho\sin(\theta) \longrightarrow b = 2\sin\left(\frac{\pi}{4}\right) = 2\frac{\sqrt{2}}{2} = \sqrt{2}$

$$z_1 = \sqrt{2} + \sqrt{2}i$$

2. $$z_2 = \sqrt{3}e^{i\frac{\pi}{2}}$$

- $a = \rho\cos(\theta) \longrightarrow a = \sqrt{3}\cos\left(\frac{\pi}{2}\right) = \sqrt{3}\cdot 0 = 0$
- $b = \rho\sin(\theta) \longrightarrow b = \sqrt{3}\sin\left(\frac{\pi}{2}\right) = \sqrt{3}\cdot 1 = \sqrt{3}$

$$z_2 = \sqrt{3}i$$

# SOMMA

# Forma algebrica: somma

$$z_1 = a + ib$$
$$z_2 = c + id$$

## Somma

$$z_1 + z_2 = a + ib + c + id = (a + c) + i(b + d)$$

$$\mathbf{Re}(z_1 + z_2) = \mathbf{Re}(z_1) + \mathbf{Re}(z_2)$$

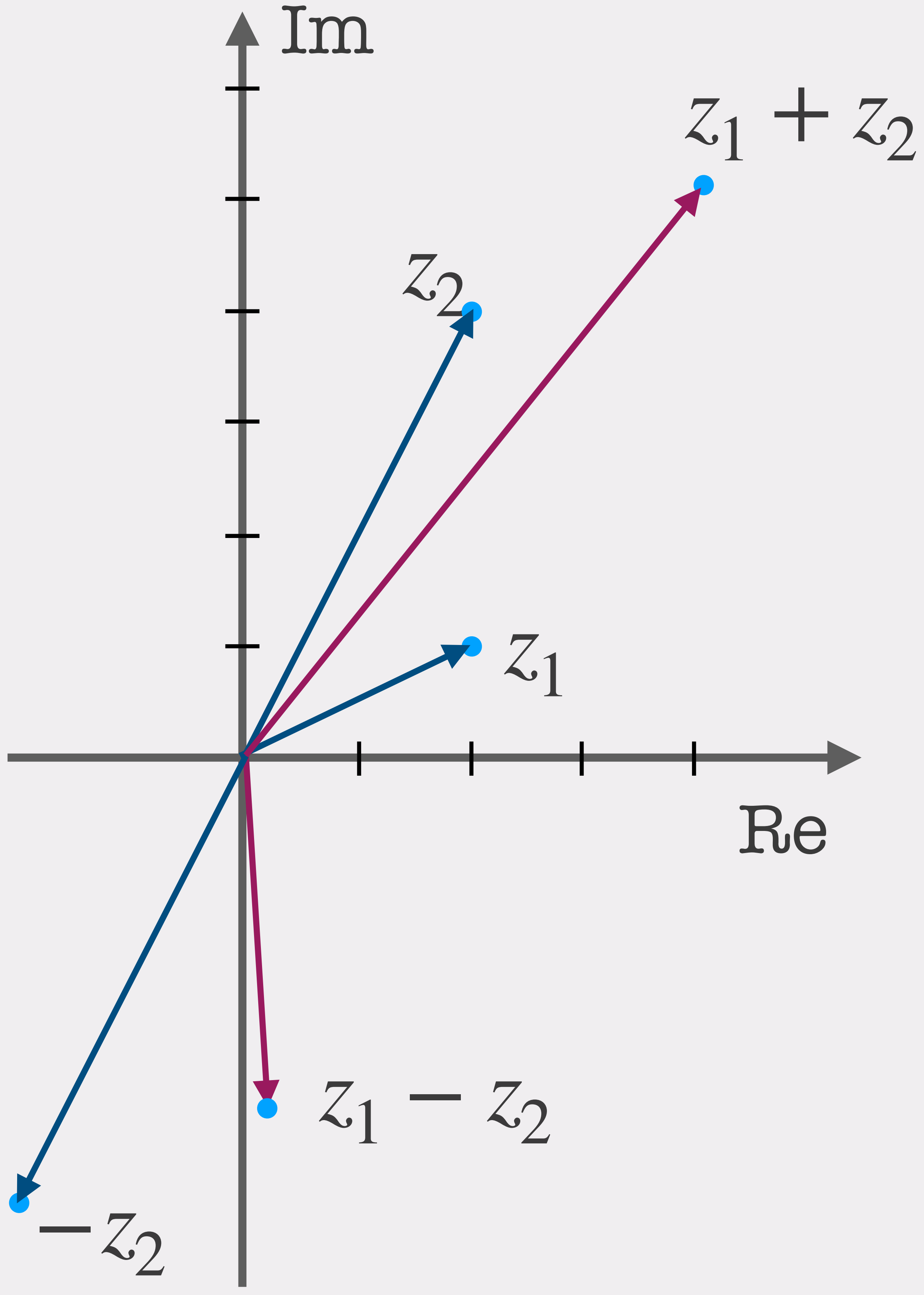
$$\mathbf{Im}(z_1 + z_2) = \mathbf{Im}(z_1) + \mathbf{Im}(z_2)$$

## Differenza

$$z_1 - z_2 = a + ib - (c + id) = (a - c) + i(b - d)$$

$$\mathbf{Re}(z_1 - z_2) = \mathbf{Re}(z_1) - \mathbf{Re}(z_2)$$

$$\mathbf{Im}(z_1 - z_2) = \mathbf{Im}(z_1) - \mathbf{Im}(z_2)$$



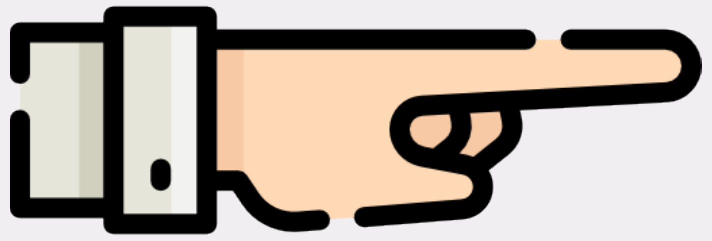
# Forma algebrica: somma

1. $z_1 = 3 - i$, $z_2 = -2 + 2i$

$$z_1 + z_2 = 3 - i + (-2 + 2i) = 3 - i - 2 + 2i = 1 + i$$

$$z_1 - z_2 = 3 - i - (-2 + 2i) = 3 - i + 2 - 2i = 5 - 3i$$

$$z_2 - z_1 = -2 + 2i - (3 - i) = -2 + 2i - 3 + i = -5 + 3i$$



PROVA TU!

2. $z_1 = 6 + 2i$, $z_2 = -3 + i$

$$z_1 + z_2 = ?$$

$$z_1 - z_2 = ?$$

$$z_2 - z_1 = ?$$

# Forma trigonometrica: somma

$$z_1 = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

$$z_2 = 4\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

$$z_1 + z_2 = 2\cos\left(\frac{\pi}{3}\right) + 4\cos\left(\frac{\pi}{6}\right) + i\left[2\sin\left(\frac{\pi}{3}\right) + 4\sin\left(\frac{\pi}{6}\right)\right] =$$

$$= 2\left(\frac{1}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right) + i\left[2\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right)\right] =$$

$$= 1 + 2\sqrt{3} + i\left(\sqrt{3} + 2\right)$$

# Forma esponenziale: somma

$$z_1 = 2e^{i\frac{\pi}{6}}$$

$$z_2 = \sqrt{3}e^{i\frac{3\pi}{2}}$$

$$z_1 + z_2 = 2e^{i\frac{\pi}{6}} + \sqrt{3}e^{i\frac{3\pi}{2}}$$

**Trasformare in forma geometrica**

# PRODOTTO

# Forma algebrica: prodotto

$$z_1 = a + ib$$
$$z_2 = c + id$$

**Prodotto**

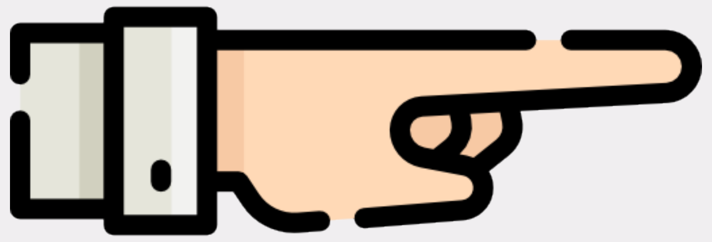
$$z_1 \cdot z_2 = (a+ib)\cdot(c+id) = ac + iad + ibc + bdi^2 =$$
$$= ac + iad + ibc - bd =$$
$$= (ac - bd) + i(ad + bc)$$

$$\mathbf{Re}(z_1 \cdot z_2) \neq \mathbf{Re}(z_1)\cdot\mathbf{Re}(z_2)$$

$$\mathbf{Im}(z_1 \cdot z_2) \neq \mathbf{Im}(z_1)\cdot\mathbf{Im}(z_2)$$

1. $z_1 = 3 - i$, $z_2 = -2 + 2i$ $\longrightarrow$

$$z_1 \cdot z_2 = (3-i)\cdot(-2+2i) = 3(-2) + 3\cdot 2i + (-i)(-2) + (-i)(2i) =$$
$$= -6 + 6i + 2i - 2i^2 = -4 + 8i$$



PROVA TU!

2. $z_1 = 6 + 2i$, $z_2 = -3 + i$ $\longrightarrow$ $z_1 \cdot z_2 = ?$

# Forma esponenziale: prodotto

$z_1 = \rho_1 e^{i\theta_1}$ , $z_2 = \rho_2 e^{i\theta_2}$

Prodotto

$$z_1 \cdot z_2 = \rho_1 e^{i\theta_1} \cdot \rho_2 e^{i\theta_2} = \rho_1 \cdot \rho_2 e^{i(\theta_1+\theta_2)}$$

# Forma trigonometrica: prodotto

$$z_1 = \rho_1 \left[\cos(\theta_1) + i\sin(\theta_1)\right]$$

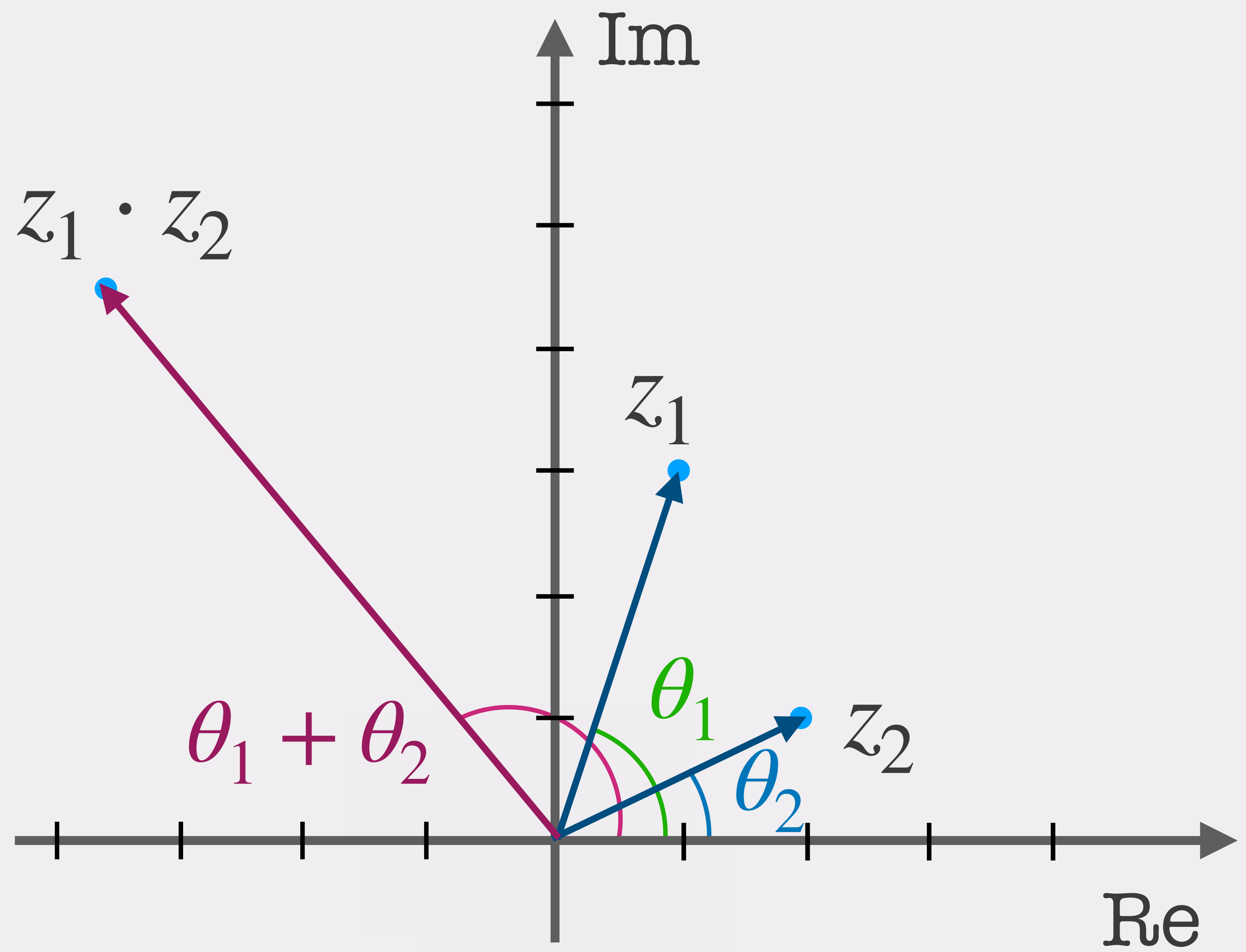
$$z_2 = \rho_2 \left[\cos(\theta_2) + i\sin(\theta_2)\right]$$

Prodotto

$$z_1 \cdot z_2 = \rho_1\rho_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

$$arg(z_1 \cdot z_2) = arg(z_1) + arg(z_2)$$



# Forma trigonometrica: prodotto

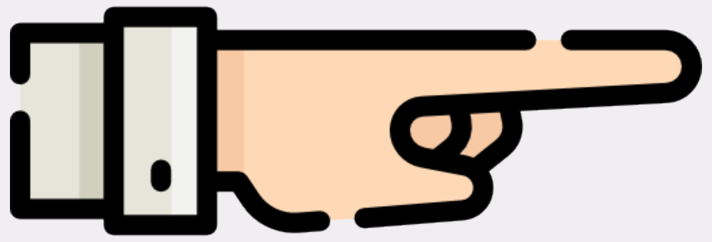
1

$$z_1 = 12\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

$$z_2 = 7\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$

$$\longrightarrow z_1 \cdot z_2 = 12 \cdot 7\left[\cos\left(\frac{\pi}{4} + \frac{5\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{5\pi}{4}\right)\right] =$$

$$= 84\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right]$$



PROVA TU!

2

$$z_1 = 3\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

$$z_2 = 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

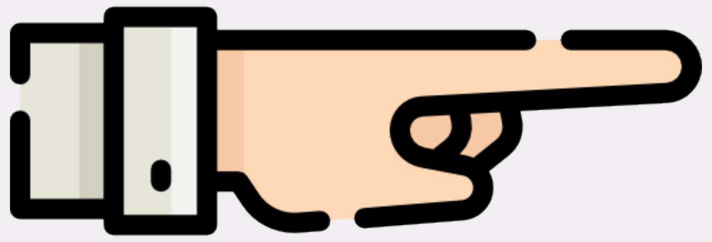
$$\longrightarrow z_1 \cdot z_2 = ?$$

# Forma esponenziale: prodotto

1

$$z_1 = \frac{1}{2}e^{i\frac{\pi}{2}} \qquad z_2 = 5e^{i\frac{\pi}{3}}$$

$$\longrightarrow \quad z_1 \cdot z_2 = \frac{1}{2} \cdot 5\ e^{i\left(\frac{\pi}{2}+\frac{\pi}{3}\right)} = \frac{5}{2}\ e^{i\frac{5\pi}{6}}$$



PROVA TU!

2

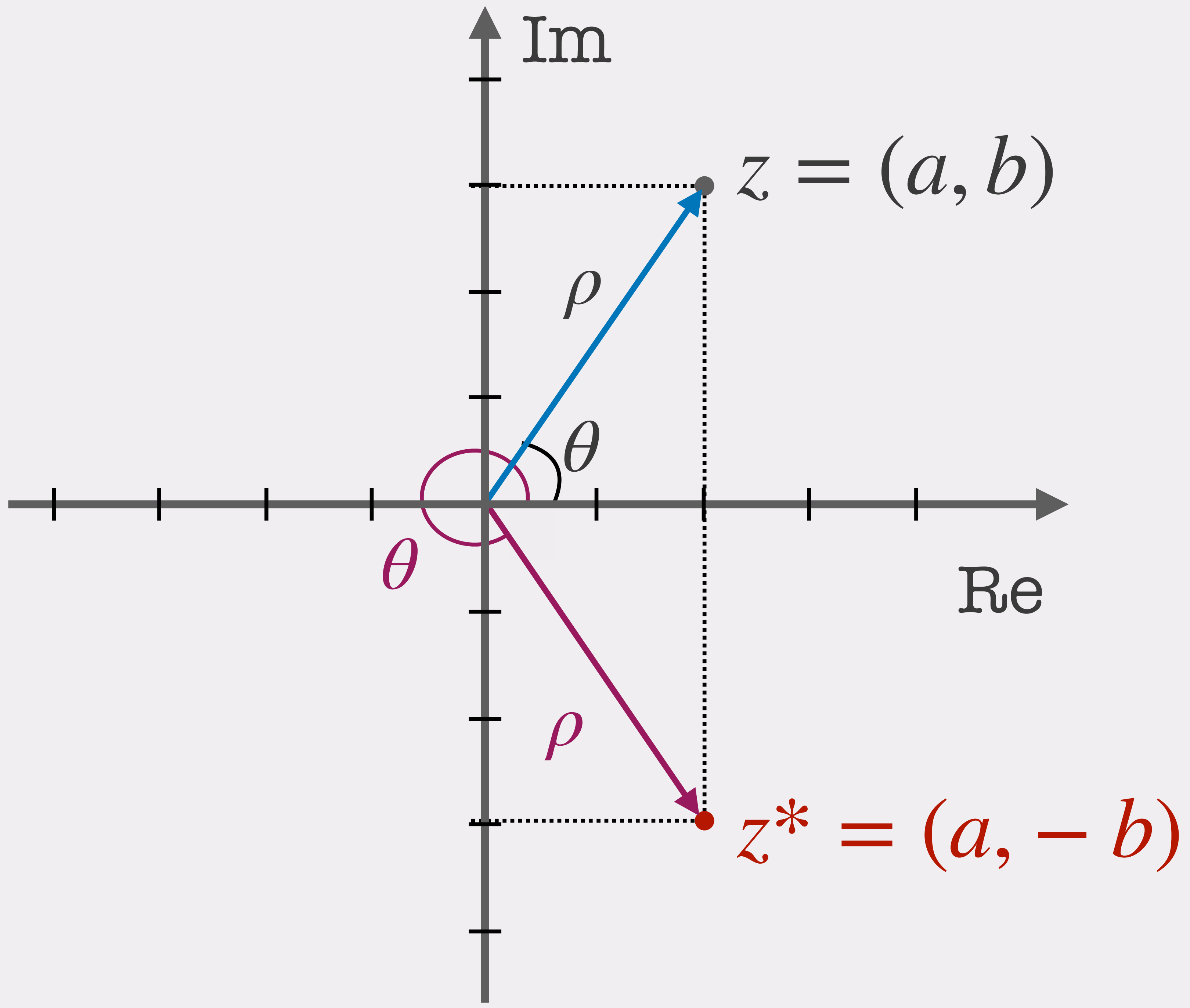
$$z_1 = 2e^{i\pi} \qquad z_2 = 3e^{i\frac{5\pi}{6}}$$

$$\longrightarrow \quad z_1 \cdot z_2 = \ ?$$

# Il **coniugato** di un numero complesso

$$\mathbf{Re}(z^*) = \mathbf{Re}(z)$$
$$\mathbf{Im}(z^*) = -\,\mathbf{Im}(z)$$

$$|z^*| = |z|$$
$$arg(z^*) = 2\pi - arg(z)$$




Forma algebrica:

$$z = a + ib \longrightarrow z^* = a - ib$$

Forma trigonometrica :

$$z = \rho\Big[\cos(\theta) + i\sin(\theta)\Big] \longrightarrow z^* = \rho\Big[\cos(2\pi - \theta) + i\sin(2\pi - \theta)\Big]$$

Forma esponenziale :

$$z = \rho e^{i\theta} \longrightarrow z^* = \rho e^{i(2\pi - \theta)}$$

1 $z_1 = -\sqrt{3} + \frac{2}{3}i \longrightarrow z_1^* = -\sqrt{3} - \frac{2}{3}i$

2 $z_2 = 7\ e^{i\frac{\pi}{3}} \longrightarrow z_2^* = 7\ e^{i\left(2\pi - \frac{\pi}{3}\right)} = 7\ e^{i\frac{5\pi}{3}}$

3 $z_3 = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$

$\longrightarrow z_3^* = \sqrt{2}\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]$

1 $z + z^* = 2\mathbf{Re}(z)$

$$z = a + ib \quad z^* = a - ib \longrightarrow z + z^* = a + ib + a - ib = 2a$$

2 $z - z^* = 2i\mathbf{Im}(z)$

$$z = a + ib \quad z^* = a - ib \longrightarrow z - z^* = a + ib - (a - ib) = 2ib$$

3 $z \cdot z^* = |z|^2$

$$z = \rho e^{i\theta} \quad z^* = \rho e^{i(2\pi - \theta)} \longrightarrow z \cdot z^* = \rho e^{i\theta} \cdot \rho e^{i(2\pi-\theta)} = \rho^2 e^{i2\pi}$$

$$= \rho^2\left[\cos(2\pi) + i\sin(2\pi)\right] = \rho^2$$